Coherence of neutrino flavor mixing in quantum field theory

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(September 1999)

In the simplistic quantum mechanical picture of flavor mixing, conditions on the maximum size and minimum coherence time of the source and detector regions for the observation of interference—as well as the very viability of the approach—can only be argued in an ad hoc way from principles external to the formalism itself. To examine these conditions in a more fundamental way, the quantum field theoretical S-matrix approach is employed in this paper, without the unrealistic assumption of microscopic stationarity. The fully normalized, time-dependent neutrino flavor mixing event rates presented here automatically reveal the coherence conditions in a natural, self-contained, and physically unambiguous way, while quantitatively describing the transition to their failure.

14.60.Pq, 26.65.+t, 13.15.+g

I. INTRODUCTION

Several recent works have examined neutrino flavor mixing by considering the neutrino production/mixing/detection as a single process in the context of quantum field theory (QFT) [1–8]. Such a framework clarifies several conceptual difficulties associated with the familiar quantum mechanical (QM) model of the flavor mixing process (see Ref. [9] for a listing of some of these). One conceptual difficulty associated with the simplified QM picture is that it postulates neutrino flavor eigenstates of indefinite mass, while in QFT external particles (asymptotic states) are generally required to be on-shell. Hence the usual methods of calculating neutrino production rates in QFT would be rates for neutrinos of a particular mass, precluding interference between neutrino states of different mass. In a QFT description of a neutrino mixing experiment, this problem is resolved by considering the neutrinos to be virtual particles. After all, it is the measurable, on-shell external charged leptons associated with the neutrino production and detection processes that operationally define what is meant by "neutrino flavor mixing"; the neutrinos themselves are not directly observed. In the relativistic limit, the same factors that constitute the "oscillation amplitude" in the simplified quantum mechanical picture can be identified in the amplitude for the overall neutrino production/mixing/detection process.

While descriptions of neutrino flavor mixing in QFT have provided insight, some shortcomings remain. As noted in Ref. [8], one problem is that the calculations [1–7] are not carried out to normalized event rates. Without normalization, one cannot definitely say that one has identified an "oscillation probability."[1] In addition, Refs. [1–3,5,6] are restricted to particular neutrino production and detection reactions, while Ref. [4] employs idealized two-state systems as source and detector. Since one would hope to justify the use of the simple QM model in general circumstances, such restrictions should not be required.

Another potential pitfall is a failure to distinguish between macroscopic stationarity and microscopic stationarity.[2] Some previous studies invoke microscopic stationarity, either implicitly [3,7,8], or with explicit reference to bound states [2,6] in the source and/or detector. While sources and detectors as a whole can sometimes be considered stationary on a macroscopic basis, the claim that individual particles in the source and/or detector remain unperturbed in coherent states over macroscopic time scales is dubious.[3] A good example is the Sun: While macroscopic variables such as density, pressure, and so on may be stationary, zooming in to atomic scales one sees a roiling thermodynamic bath of particles being created, destroyed, and scattered on rapid time scales. Clearly there is no hope of appealing

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[1]A normalized probability is given in Ref. [7], but it is not the experimentally relevant one, which is identified by making a complete connection of the squared amplitude for the production/mixing/detection process with the form (neutrino flux)(oscillation probability) (neutrino cross section) [8].

[2]This critique applies only to QFT analyses of mixing experiments, which by their very nature purport to describe microscopic processes in the source and detector. Once the feasibility of the use of flavor eigenstates in a simplified model is established, macroscopic stationarity can be sensibly employed, as in Ref. [10].

[3]The "macroscopic time scale" at issue here is the signal travel time between source and detector. In the case of astrophysical neutrino sources, this time is macroscopic indeed.

to bound states of the nuclei and electrons which collide to produce neutrinos in the Sun! Even in detectors which have bound state particles, it is difficult to conceive of these states as being coherent over macroscopic time scales. For example, water Čerenkov detectors see charged lepton wave packets with finite energy and time spread (after all, an "event time" is recorded whose uncertainty is much smaller than, for example, the signal travel time between the Sun and Earth). While the overall chain of the processes of detection can be rather complex, one would expect that at least part of the reason for the finite spread of these detected charged leptons is the limited coherence time of the bound state particles with which the initial detection interaction takes place.

In short, if one employs a QFT description of flavor oscillations in order to overcome the conceptual difficulties of the simplified QM model, one should also pay the price of being realistic about the lack of microscopic stationarity in order to complete a convincing picture. In this paper, this is accomplished by treating all of the initial and final state external particles as wave packets which have finite overlap in space and time in the source and detector. Thus it is similar in spirit to Refs. [1,5], but in pursuit of generality the treatment does not specify particular neutrino production and detection mechanisms or specific functional forms for the wave packets of the external particles involved in the production and detection processes. Another difference of the present treatment is a greater emphasis on the coordinate space Green's function, as in Ref. [8], where it was used to make direct contact with the standard simple coordinate space formalism for the MSW effect. In this work no integration is performed over the time coordinate of the detection event, since many experiments—including those employing water Čerenkov detectors—record this time. (An integration *is* performed over the unobserved time coordinate of the emission event.) Finally, the detailed connection of the squared amplitude for the microscopic neutrino production/mixing/detection process to macroscopic event rates will be made. Only this complete connection—with all factors accounted for—enables one to define an oscillation probability.

II. S-MATRIX APPROACH TO NEUTRINO MIXING PROCESSES

A common application of QFT is the description of particle collisions in accelerators. Rates or cross sections associated with these processes can be obtained in a heuristic manner directly from the plane wave scattering S-matrix computed from Feynman diagrams,

$$S(\{p\}) - 1 \equiv (2\pi)^4 \delta^4 \left(\sum_l (-1)^{d_l} p_l \right) i\mathcal{M}(\{p\}). \tag{1}$$

In this expression $\{p\}$ is the set of external particle momenta p_l, $d_l = 1$ for incoming and 0 for outgoing particles, and $\mathcal{M}$ is the δ function-free matrix element. The event rate obtained from Eq. (1) is[4]

$$d\Gamma = (2\pi)^4 \delta^4 \left(\sum_l (-1)^{d_l} p_l \right) V^{1-I} |\mathcal{M}(\{p\})|^2 \left[\prod_i^I \frac{1}{2E_{\mathbf{p}_i}} \right] \left[\prod_{i'}^F \frac{d\mathbf{p}_{i'}}{(2\pi)^3 (2E_{\mathbf{p}_{i'}})} \right]. \tag{2}$$

Here V is the three-volume in which the entire process occurs, I and F are the numbers of particles in the initial and final states, and the components of the on-shell four-momenta p_i are $(E_{\mathbf{p}_i}, \mathbf{p}_i)$. This mnemonic for arriving at event rates is possible because the interactions of interest occur in a single, small spacetime volume. It is more convincingly justified, however, by a wave packet description (e.g., Ref. [12]).

One of the reasons one considers a QFT description of neutrino flavor mixing is that the standard picture of requiring external particles (asymptotic states) to be on-shell precludes the existence of massive neutrino flavor eigenstates (assuming that each charged lepton couples to multiple neutrino fields of different masses). Accordingly, one considers the neutrinos as virtual particles in a Feynman diagram in which the charged leptons at the source and detection vertices identify the neutrino flavor. Neutrino flavor mixing then results from interference of diagrams whose intermediate neutrinos have different masses.

In this picture it is not possible to compute event rates directly from the S-matrix with the usual mnemonic described above. This is because a neutrino oscillation experiment involves neutrino production and detection regions which are widely separated in space. In contrast to the case of accelerator particle collisions, the interactions of interest do not all occur in a single volume element. In addition, as argued in Sec. I, in this microscopic picture the production and detection of a single neutrino will be separated in time as well as space.

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[4] The conventions for the metric, γ matrices, and normalizations employed here are the same as those of Ref. [11].

In order to describe the spacetime localization one must fall back on a wave packet description of the external particles, in which the amplitude is a superposition of plane wave amplitudes:

$$\mathcal{A} = \int \prod_j^{1+F_D} [dp_j] \; \phi_{Dj}(p_j, \underline{\mathbf{p}}_j) \; \prod_i^{I_S+F_S} [dk_i] \; \phi_{Si}(k_i, \underline{\mathbf{k}}_i) \; [S(\{k\},\{p\}) - 1] , \tag{3}$$

where (for example) $[dp_j] = d\mathbf{p}_j / \left[(2\pi)^3 \sqrt{2E_{\mathbf{p}_j}}\right]$, $\{k\}$ are the external momenta connected to the vertex causing neutrino production, and $\{p\}$ are the external momenta connected to the vertex associated with neutrino detection. The quantities $\{\underline{\mathbf{k}}\}$ and $\{\underline{\mathbf{p}}\}$ denote the peak of the wave packets' distribution of three-momenta. There are I_S incoming and F_S outgoing particles connected to the production vertex, and 1 incoming and F_D outgoing external particles at the detection vertex. The origin of the source wave packets is taken to be the spacetime point x_S:

$$\phi_{Si}(k_i, \underline{\mathbf{k}}_i) = a_{Si}(\mathbf{k}_i - \underline{\mathbf{k}}_i)\, e^{-i(-1)^{d_i} k_i \cdot x_S}, \tag{4}$$

and similarly for the detector wave packets with origin y_D. The real function $a(\mathbf{k} - \underline{\mathbf{k}})$ is peaked about $\underline{\mathbf{k}}$. In order that Eq. (3) describe the amplitude for interaction of one localized external particle of each type, the wave packet normalization must be, e.g. for the source packets,

$$\int \frac{d\mathbf{k}_i}{(2\pi)^3} \left|\phi_{Si}(k_i, \underline{\mathbf{k}}_i)\right|^2 \; = \int \frac{d\mathbf{k}_i}{(2\pi)^3} \left|a_{Si}(\mathbf{k}_i - \underline{\mathbf{k}}_i)\right|^2 = \; 1. \tag{5}$$

It is convenient to define the transform

$$\psi_{\underline{\mathbf{k}}_i}(\mathbf{x}) = \int \frac{d\mathbf{k}_i}{(2\pi)^3}\, a_{Si}(\mathbf{k}_i - \underline{\mathbf{k}}_i)\, e^{-i(-1)^{s_i} \mathbf{k} \cdot \mathbf{x}}, \tag{6}$$

where s_i behaves the same as the d_l of Eq. (1). The normalization of this function is

$$\int d\mathbf{x} \left|\psi_{\underline{\mathbf{k}}_i}(\mathbf{x})\right|^2 = 1, \tag{7}$$

which follows from Eq. (5).

The next step is to transform the momentum-based Eq. (3) into a coordinate space expression. The S-matrix in Eq. (3) can be expressed

$$\begin{aligned} S(\{k\},\{p\}) - 1 = & \int d^4y \; e^{i \sum_l (-1)^{d_l} p_l \cdot y} \int d^4x \; e^{i \sum_l (-1)^{s_l} k_l \cdot x} \\ & \times i \int \frac{d^4 s}{(2\pi)^4} e^{\mp i s \cdot (y - x)} M_2 \, P_L \, G(s) \, P_R \, M_1, \end{aligned} \tag{8}$$

in which s is the off-shell neutrino propagator momentum. The upper (lower) sign of $\mp$ in the exponential is for neutrino (antineutrino) mixing. This arises from choosing x (y) to always correspond to the source (detector). That is, for neutrino oscillations of flavor α to flavor β, the propagator is $iG^{\beta\alpha}(y,x) = \langle T\{\nu^\beta(y)\bar{\nu}^\alpha(x)\}\rangle_0 = i\int d^4s\,(2\pi)^{-4}\, e^{-is\cdot(y-x)} G^{\beta\alpha}(s)$ (with $T\{\}$ and $\langle\rangle_0$ denoting a time-ordered product and vacuum expectation value respectively), while for antineutrino oscillations $\alpha \to \beta$, the labeling is $iG^{\alpha\beta}(x,y)$. $V - A$ interactions have been assumed; P_L and P_R are the left- and right-handed projection operators, with M_1 and M_2 column and row vectors in spinor space respectively. For neutrino mixing, $M_1 = M_1(\{k\})$ and $M_2 = M_2(\{p\})$ are respectively associated with the neutrino production and detection reactions. For antineutrino mixing, $M_2 = M_2(\{k\})$ and $M_1 = M_1(\{p\})$ are respectively associated with the production and detection reactions. The partially transformed propagator $G(s^0, \mathbf{y}, \mathbf{x})$ is defined by

$$\int \frac{d^4 s}{(2\pi)^4}\, e^{\mp i s \cdot (y-x)} G(s) = \int \frac{ds^0}{2\pi}\, e^{\mp i s^0 (y^0 - x^0)}\, G(s^0, \mathbf{y}, \mathbf{x}). \tag{9}$$

It is assumed that the wave packets $a(\mathbf{k} - \underline{\mathbf{k}})$ are sufficiently well-peaked that integrals of the following form can be evaluated in an approximate manner:

$$\begin{aligned} \int [dk_i]\, \phi_{Si}(k_i, \underline{\mathbf{k}}_i)\, e^{i(-1)^{s_i} k_i \cdot x} M(\mathbf{k}_i) = & \int \frac{d\mathbf{k}_i}{(2\pi)^3 \sqrt{2E_{\mathbf{k}_i}}}\, a_{Si}(\mathbf{k}_i - \underline{\mathbf{k}}_i)\, e^{i(-1)^{s_i} k_i \cdot (x - x_S)} M(\mathbf{k}_i) \\ \approx & \; \frac{e^{i(-1)^{s_i} \underline{k}_i \cdot (x - x_S)}}{\sqrt{2E_{\underline{\mathbf{k}}_i}}}\; \psi_{\underline{\mathbf{k}}_i}\left((\mathbf{x} - \mathbf{x}_S) - (x^0 - x^0_S)\mathbf{v}_{\underline{\mathbf{k}}_i}\right)\, M(\underline{\mathbf{k}}_i). \end{aligned} \tag{10}$$

In Eq. (10), $\mathbf{v}_{\underline{\mathbf{k}}_i}$ is the wave packet's group velocity $(\nabla_{\mathbf{k}_i} k_i^0)|_{\mathbf{k}_i=\underline{\mathbf{k}}_i} = \underline{\mathbf{k}}_i / E_{\underline{\mathbf{k}}_i}$, and wave packet spreading has been neglected. Similar expressions hold for the detector wave packets. With this approximation, the amplitude of Eq. (3) becomes

$$\mathcal{A} = \left(\prod_i^{I_S+F_S} \frac{1}{\sqrt{2E_{\underline{\mathbf{k}}_i}}}\right) \left(\prod_j^{1+F_D} \frac{1}{\sqrt{2E_{\underline{\mathbf{p}}_j}}}\right) \int d^4x\, d^4y\, e^{i\sum_l (-1)^{s_l} \underline{k}_l \cdot (x-x_S)} e^{i\sum_l (-1)^{d_l} \underline{p}_l \cdot (y-y_D)}$$
$$\times \left[\prod_i^{I_S+F_S} \psi_{\underline{\mathbf{k}}_i}\left((\mathbf{x}-\mathbf{x}_S)-(x^0-x_S^0)\mathbf{v}_{\underline{\mathbf{k}}_i}\right)\right] \left[\prod_j^{1+F_D} \psi_{\underline{\mathbf{p}}_j}\left((\mathbf{y}-\mathbf{y}_D)-(y^0-y_D^0)\mathbf{v}_{\underline{\mathbf{p}}_j}\right)\right]$$
$$\times i \int \frac{ds^0}{2\pi}\, e^{\mp i s^0 (y^0-x^0)}\, \underline{M}_2 P_L\, G(s^0, \mathbf{y}, \mathbf{x})\, P_R \underline{M}_1, \tag{11}$$

in which the bars in $\underline{M}_1$ and $\underline{M}_2$ signify that these quantities have been evaluated at the peak momenta of the wave packets.

III. APPROXIMATION OF THE WAVE PACKET OVERLAP

It is convenient at this stage to adopt an approximation regarding the overlap of the wave packets that captures the essential physics while maintaining mathematical simplicity. The initial and final state wave packets in the source (for example), traveling with their various group velocities, overlap in a limited region of space for a limited time. To give a specific definition to this spacetime volume of the overlap, $\mathcal{V}_S$, centered on x_S, it is convenient to define

$$E_S(x-x_S) \equiv \left[\prod_i^{I_S+F_S} \psi_{\underline{\mathbf{k}}_i}\left(\mathbf{x}_S, x_S^0\right)\right]^{-1} \left[\prod_i^{I_S+F_S} \psi_{\underline{\mathbf{k}}_i}\left((\mathbf{x}-\mathbf{x}_S)-(x^0-x_S^0)\mathbf{v}_{\underline{\mathbf{k}}_i}\right)\right], \tag{12}$$

where the notation

$$\psi_{\underline{\mathbf{k}}_i}\left(\mathbf{x}_S, x_S^0\right) = \psi_{\underline{\mathbf{k}}_i}\left((\mathbf{x}-\mathbf{x}_S)-(x^0-x_S^0)\mathbf{v}_{\underline{\mathbf{k}}_i}\right)\Big|_{\mathbf{x}=\mathbf{x}_S, x^0=x_S^0} \tag{13}$$

has been adopted in the first factor. Then $\mathcal{V}_S$ is defined by

$$\int d^4x\, [E_S(x-x_S)]^2 = \int d^4x \exp\left\{2\ln\left[1-\frac{1}{2}(W_S)_{\mu\nu}(x-x_S)^\mu (x-x_S)^\nu + \cdots\right]\right\}$$
$$\approx \frac{\pi^2}{\sqrt{\mathrm{Det}\,[(W_S)_{\mu\nu}]}} \equiv \mathcal{V}_S, \tag{14}$$

where

$$(W_S)_{\mu\nu} \equiv -\frac{\partial^2}{\partial x^\mu \partial x^\nu} E_S(x-x_S)\Big|_{x=x_S}. \tag{15}$$

The timelike (spacelike) components of $(W_S)_{\mu\nu}$ reflect the spread of energy (momentum) available in the reaction, while the timelike (spacelike) components of $(W_S^{-1})_{\mu\nu}$ characterize the extent in time (space) of the wave packet overlap. Similar considerations apply to the detector region.

It is only necessary here to consider the Green's function for neutrino propagation through the vacuum. Inspection of Ref. [8] indicates that the vacuum propagator results will be applicable in a relatively direct way to the case of neutrino propagation through a medium of constant density. Generalization to the case of a medium of varying density would be more complicated, however. While interference terms for this case have been calculated in the context of the simple quantum mechanical model [13], they are not relevant to current observations of astrophysical neutrinos. Hence the effort to study the microscopic origin for the damping of interference terms already deemed irrelevant does not seem to be worthwhile at present.

Focusing on the vacuum case—for which the interference terms *are* of current experimental interest—the final factors in Eq. (11) can be expressed as

$$\underline{M}_2 P_L\, G(s^0, \mathbf{y}, \mathbf{x})\, P_R \underline{M}_1 = \underline{\tilde{M}}_2 \tilde{G}(s^0, \mathbf{y}, \mathbf{x}) \underline{\tilde{M}}_1, \tag{16}$$

where $\underline{\tilde{M}}_2$ and $\underline{\tilde{M}}_1$ are respectively the two-component subspinors that remain nonzero in $\underline{M}_2 P_L$ and $P_R \underline{M}_1$, and $\tilde{G}(s^0, \mathbf{y}, \mathbf{x})$ is the nonzero 2×2 submatrix in $P_L G(s^0, \mathbf{y}, \mathbf{x}) P_R$. Because the overlaps of the wave packets are restricted to the vicinity of $\mathbf{x}_S$ and $\mathbf{y}_D$, and because $|\mathbf{y}_D - \mathbf{x}_S| \gg L_S, L_D$, the leading contribution from the Green's function is of the form [8]

$$\tilde{G}^{\alpha\beta}(s^0, \mathbf{y}, \mathbf{x}) \approx -\sum_k U_{\alpha k} U^*_{\beta k} \left(s^0 - \frac{s^0}{|s^0|}\, s_k\, \boldsymbol{\sigma} \cdot \hat{\mathbf{L}} \right) \frac{e^{i s_k \hat{\mathbf{L}} \cdot (\mathbf{y} - \mathbf{x})}}{4\pi |\mathbf{y}_D - \mathbf{x}_S|}. \tag{17}$$

In this expression, the flavor and mass fields are related by $\nu_\alpha(x) = \sum_k U_{\alpha k} \psi_k(x)$; $s_k = \sqrt{(s^0)^2 - m_k^2}$, in which m_k is the mass associated with the neutrino field $\psi_k(x)$; $\boldsymbol{\sigma}$ is the three-vector of Pauli matrices; and the vector $\hat{\mathbf{L}} = (\mathbf{y}_D - \mathbf{x}_S)/|\mathbf{y}_D - \mathbf{x}_S|$ points from the source to the detector. For neutrino oscillations, β is the flavor of the charged lepton associated with the source reaction, and α is the flavor of the charged lepton associated with the detection reaction. For antineutrino oscillations these assignments are reversed.

With these preparations the remaining integrations in Eq. (11) can be performed. Employing Eqs. (12), (16), and (17), and employing a similar approximation to that employed in Eq. (14) for the x and y integrals, the amplitude becomes

$$\begin{aligned}\mathcal{A} = -i &\left[\prod_i^{I_S + F_S} \frac{\psi_{\underline{\mathbf{k}}_i}\left(\mathbf{x}_S, x_S^0\right)}{\sqrt{2E_{\underline{\mathbf{k}}_i}}} \right] \left[\prod_j^{1 + F_D} \frac{\psi_{\underline{\mathbf{p}}_j}\left(\mathbf{y}_D, y_D^0\right)}{\sqrt{2E_{\underline{\mathbf{p}}_j}}} \right] \frac{(4\mathcal{V}_S)(4\mathcal{V}_D)}{4\pi |\mathbf{y}_D - \mathbf{x}_S|} \\ &\times \sum_k U_{\alpha k} U^*_{\beta k} \int \frac{ds^0}{2\pi}\, e^{\mp i s^0 (y_D^0 - x_S^0) + i s_k |\mathbf{y}_D - \mathbf{x}_S| - D_k(s^0)}\, \underline{\tilde{M}}_2 \left(s^0 - \frac{s^0}{|s^0|}\, s_k\, \boldsymbol{\sigma} \cdot \hat{\mathbf{L}} \right) \underline{\tilde{M}}_1,\end{aligned} \tag{18}$$

in which the function $\exp[-D_k(s^0)]$, with

$$\begin{aligned}D_k(s^0) = &\frac{1}{2} (W_S^{-1})_{\mu\nu} \left(-k_S + \xi_k\right)^\mu \left(-k_S + \xi_k\right)^\nu \\ &+ \frac{1}{2} (W_D^{-1})_{\mu\nu} \left(p_D - \xi_k\right)^\mu \left(p_D - \xi_k\right)^\nu\end{aligned} \tag{19}$$

enforces energy-momentum conservation to the extent allowed by the finite overlap in space and time of the external particle wave packets. In Eq. (19), the notation

$$k_S \equiv -\sum_l (-1)^{s_l} \underline{k}_l, \tag{20}$$

$$p_D \equiv +\sum_l (-1)^{d_l} \underline{p}_l, \tag{21}$$

$$\xi_k \equiv \left(\pm s^0, \hat{\mathbf{L}} \sqrt{(s^0)^2 - m_k^2} \right) \tag{22}$$

has been employed.

If plane wave final states and stationary initial source and/or detector particle states had been employed, the finite energy spread indicated in Eq. (19) would have been replaced by an energy delta function, suggesting the idea that the neutrinos are energy eigenstates. In such a case, one finds $s^0 = \pm p_D^0 = \pm k_S^0$ (it will be recalled that the upper sign is for neutrino emission at the source, and the lower sign for antineutrino emission). It is easy to see by considering sample neutrino production and detection processes that k_S^0 and p_D^0 should be positive quantities.

While Eq. (19) indicates that a range of s^0 contributes to the amplitude, there is a value of s^0—call it $(\underline{s}^0)_k$—which makes the largest contribution to the amplitude. This is the value of s^0 for which $D_k(s^0)$ has its minimum value. The relative degrees to which overall energy and momentum are conserved compete in determining $(\underline{s}^0)_k$. Since the external particles travel at speeds less than the speed of light, however, the timelike components of the tensors $(W_{S,D}^{-1})_{\mu\nu}$ will be larger than the spacelike components. This means that, in analogy with the stationary situation mentioned above, $(\underline{s}^0)_k$ may be taken to be positive (negative) for neutrino (antineutrino) emission at the source. This, together with the fact that only the region in the vicinity of $(\underline{s}^0)_k$ will be taken into account in the approximate evaluation of the integral, means that the integral over s^0 can be replaced by integration over a new variable λ, with $\pm s^0$ replaced by λ in the integrand. This new integral is dominated by the region near $\underline{\lambda}_k \equiv |(\underline{s}^0)_k|$, determined by

$$0 = \frac{dD_k(\lambda)}{d\lambda}, \tag{23}$$

in which $D_k(\lambda)$ is given by Eq. (19) with $\xi_k = \left(\lambda, \hat{\mathbf{L}}\sqrt{\lambda^2 - m_k^2}\right)$. Expanding $D_k(\lambda)$ to second order about $\underline{\lambda}_k$, the rest of the argument of the exponential to first order, the rest of the integrand to zeroth order, performing the λ integration, and squaring the amplitude yields the result

$$
\begin{aligned}
|\mathcal{A}|^2 = & \left[\prod_i^{I_S+F_S} \frac{\left|\psi_{\underline{\mathbf{k}}_i}\left(\mathbf{x}_S, x_S^0\right)\right|^2}{\left(2E_{\underline{\mathbf{k}}_i}\right)}\right] \left[\prod_j^{1+F_D} \frac{\left|\psi_{\underline{\mathbf{p}}_j}\left(\mathbf{y}_D, y_D^0\right)\right|^2}{\left(2E_{\underline{\mathbf{p}}_j}\right)}\right] \frac{4(\mathcal{V}_S)^2(\mathcal{V}_D)^2}{\pi^4|\mathbf{y}_D - \mathbf{x}_S|^2} \left|\sum_k U_{\alpha k} U_{\beta k}^* \left(\frac{\pi}{\ell_k^2}\right)^{1/2}\right. \\
& \times \exp\left[-i\underline{\lambda}_k(y_D^0 - x_S^0) + i s_k(\underline{\lambda}_k)|\mathbf{y}_D - \mathbf{x}_S| - C_k(\underline{\lambda}_k, x_S, y_D) - D_k(\underline{\lambda}_k)\right] \\
& \left. \times\, \underline{\tilde{M}}_2 \left[\underline{\lambda}_k - s_k(\underline{\lambda}_k)\,\boldsymbol{\sigma}\cdot\hat{\mathbf{L}}\right] \underline{\tilde{M}}_1 \right|^2, \qquad (24)
\end{aligned}
$$

where

$$
\ell_k^2 = \frac{1}{2}\frac{d^2 D_k(\lambda)}{d\lambda^2}\bigg|_{\lambda=\underline{\lambda}_k}. \qquad (25)
$$

Study of the explicit expression for ℓ_k^2 shows that it is essentially the sum of the squares of the time and length scales of the wave packet overlaps in the source and detector, i.e.

$$
\ell_k^2 \ \sim\ (T_S)^2 \ +\ (L_S)^2 \ +\ (T_D)^2 \ + (L_D)^2, \qquad (26)
$$

(in rather obvious notation). This is particularly transparent in the limit of relativistic neutrinos.

The factor $\exp[-C_k(\underline{\lambda}_k, x_S, y_D)]$, with

$$
C_k(\underline{\lambda}_k, x_S, y_D) = \frac{1}{4\ell_k^2}\left[(y_D^0 - x_S^0) - \frac{1}{v_k}|\mathbf{y}_D - \mathbf{x}_S|\right]^2, \qquad (27)
$$

suppresses contributions from neutrinos that do not follow a classical spacetime trajectory between the production event at $(x_S^0, \mathbf{x}_S)$ and the detection event at $(y_D^0, \mathbf{y}_D)$ [the neutrino velocity is given by $v_k = s_k(\underline{\lambda}_k)/\underline{\lambda}_k$].[5]

IV. MACROSCOPIC EVENT RATE

To make contact with experiments it is necessary to magnify the probability of Eq. (24) up to macroscopic scales. For this purpose, the normalization in Eq. (7) suggests that (for example) $\left|\psi_{\underline{\mathbf{k}}_i}\left(\mathbf{x}_S, x_S^0\right)\right|^2$ be interpreted as the (per particle) volume density of particles with momentum $\underline{\mathbf{k}}_i$ at position $\mathbf{x}_S$ and time x_S^0, where the last two quantities are now thought of as macroscopic spacetime variables. Employing the usual statistical methods for free particles, these particle densities are taken to be $[d\underline{\mathbf{k}}_i/(2\pi)^3]\, f(\underline{\mathbf{k}}_i, \mathbf{x}_S, x_S^0)$ for initial state particles (where f is the phase space density) and $[d\underline{\mathbf{k}}_i/(2\pi)^3]$ for final state particles. In connection with the (now macroscopic) variables $\mathbf{x}_S$, x_S^0, $\mathbf{y}_D$, and y_D^0, one factor of $(\mathcal{V}_S\mathcal{V}_D)$ is interpreted as $d\mathbf{x}_S\, dx_S^0\, d\mathbf{y}_D\, dy_D^0$. At the macroscopic level, a sum over external particle spins is performed; the average over initial spins is accounted for by leaving the spin degeneracy out of the phase space distribution functions f. The expected number of events detected from neutrino interactions with the $[d\underline{\mathbf{p}}/(2\pi)^3] f(\underline{\mathbf{p}}, \mathbf{y}_D, y_D^0)$ particles of momentum $\underline{\mathbf{p}}$ in detector volume $d\mathbf{y}_D$ during time dy_D^0 resulting in final state detector particles of momentum $\{\underline{\mathbf{p}}_{j'}\}$, arising from neutrinos produced from the interaction of the set of $\{[d\underline{\mathbf{k}}_i/(2\pi)^3] f(\underline{\mathbf{k}}_i, \mathbf{x}_S, x_S^0)\}$ detector particles with momenta $\{\underline{\mathbf{k}}_i\}$ in source volume $d\mathbf{x}_S$ during time dx_S^0 resulting in final state source particles of momentum $\{\underline{\mathbf{k}}_{i'}\}$, is

[5] The above procedure in which a "neutrino energy" $\underline{\lambda}_k$ is determined only from the minimum of $D_k(\lambda)$ implicitly assumes that the phase $-\lambda(y_D^0 - x_S^0) + s_k(\lambda)|\mathbf{y}_D - \mathbf{x}_S|$ in Eq. (18) is essentially stationary over the range of λ for which $\exp[-D_k(\lambda)]$ is appreciably nonzero. It is easy to see that the quantity $\exp[-C_k(\underline{\lambda}_k, x_S, y_D)]$ enforces this very condition, so that the procedure is self-consistent.

$$dN = d\mathbf{x}_S\, dx_S^0\, d\mathbf{y}_D\, dy_D^0\, d\mathbf{K}(x_S)\, d\mathbf{K}'\, d\mathbf{P}(y_D)\, d\mathbf{P}'\, \frac{4\mathcal{V}_S\mathcal{V}_D}{\pi^4|\mathbf{y}_D-\mathbf{x}_S|^2}\sum_{\text{spins}}\Bigg|\sum_k U_{\alpha k}U^*_{\beta k}\left(\frac{\pi}{\ell_k^2}\right)^{1/2}$$
$$\times \exp\left[-i\underline{\lambda}_k(y_D^0-x_S^0)+is_k(\underline{\lambda}_k)|\mathbf{y}_D-\mathbf{x}_S|-C_k(\underline{\lambda}_k,x_S,y_D)-D_k(\underline{\lambda}_k)\right]$$
$$\times\ \underline{\tilde{M}}_2\ \left[\underline{\lambda}_k-s_k(\underline{\lambda}_k)\boldsymbol{\sigma}\cdot\hat{\mathbf{L}}\right]\underline{\tilde{M}}_1\Bigg|^2, \tag{28}$$

where the notation

$$d\mathbf{K}(x_S) = \prod_i^{I_S}\frac{d\underline{\mathbf{k}}_i}{(2\pi)^3\left(2E_{\underline{\mathbf{k}}_i}\right)}\ f\left(\underline{\mathbf{k}}_i,\mathbf{x}_S,x_S^0\right), \tag{29}$$

$$d\mathbf{K}' = \prod_{i'}^{F_S}\frac{d\underline{\mathbf{k}}_{i'}}{(2\pi)^3\left(2E_{\underline{\mathbf{k}}_{i'}}\right)}, \tag{30}$$

$$d\mathbf{P}(y_D) = \frac{d\underline{\mathbf{p}}}{(2\pi)^3\left(2E_{\underline{\mathbf{p}}}\right)}\ f\left(\underline{\mathbf{p}},\mathbf{y}_D,y_D^0\right), \tag{31}$$

$$d\mathbf{P}' = \prod_{j'}^{F_D}\frac{d\underline{\mathbf{p}}_{j'}}{(2\pi)^3\left(2E_{\underline{\mathbf{p}}_{j'}}\right)} \tag{32}$$

has been introduced for the phase space factors.

While virtually all neutrino experiments record data that sums over contributions from all initial momenta in the source and detector, all source final momenta, and all source emission times, some experiments—such as those with water Čerenkov detectors—record the detector event time and (at least some) detector final state particle momenta. Hence it would not be correct to integrate over these last quantities. Also, dividing by dy_D^0 gives an expected event rate as a function of detector time y_D^0.

In the integration over all source times x_S^0, the formalism automatically "knows" that neutrinos emitted at macroscopically different source times are not allowed to interfere coherently. Each term in the squared sum of the form $|\sum_k h(k)|^2 = \sum_k\sum_{k'} h(k)h^*(k')$ in Eq. (28) has a factor $\exp[-\mathcal{T}_{kk'}]$, where

$$\mathcal{T}_{kk'} = -i(\underline{\lambda}_k-\underline{\lambda}_{k'})(y_D^0-x_S^0)-C_k(\underline{\lambda}_k,x_S,y_D)-C_{k'}(\underline{\lambda}_{k'},x_S,y_D). \tag{33}$$

Terms with $k\neq k'$ represent quantum interference terms. For a given time of detection y_D^0, $\exp[-C_k(\underline{\lambda}_k,x_S,y_D)]$ and $\exp[-C_{k'}(\underline{\lambda}_{k'},x_S,y_D)]$ tend to pick out different emission times for $k\neq k'$. If the difference in emission times is greater than the width of $\exp[-C]$, the interference is suppressed.

The gradual loss of coherence can be expressed quantitatively. The leading contribution to terms with $k\neq k'$ comes from the interval near the average emission time

$$(\underline{x}_S^0)_{kk'} = y_D^0 - \frac{1}{\underline{v}_{kk'}}|\mathbf{y}_D-\mathbf{x}_S|, \tag{34}$$

where $C_k(\underline{\lambda}_k,x_S,y_D)+C_{k'}(\underline{\lambda}_{k'},x_S,y_D)$ has a minimum. The "average velocity" $\underline{v}_{kk'}$ is given by

$$\underline{v}_{kk'} = \frac{v_k v_{k'}\left(\ell_k^2+\ell_{k'}^2\right)}{(\ell_k^2 v_k+\ell_{k'}^2 v_{k'})}. \tag{35}$$

The portion of the argument of the exponential that depends on x_S^0—that is, $\mathcal{T}_{kk'}$ of Eq. (33)—can be expressed

$$\mathcal{T}_{kk'} = \frac{-i(\underline{\lambda}_k-\underline{\lambda}_{k'})}{\underline{v}_{kk'}}|\mathbf{y}_D-\mathbf{x}_S| - \frac{(v_k-v_{k'})^2}{4v_k^2v_{k'}^2\left(\ell_k^2v_k+\ell_{k'}^2v_{k'}\right)}|\mathbf{y}_D-\mathbf{x}_S|^2$$
$$-\frac{(\underline{\lambda}_k-\underline{\lambda}_{k'})^2\ell_k^2\ell_{k'}^2}{(\ell_k^2+\ell_{k'}^2)} - \frac{\left(\ell_k^2+\ell_{k'}^2\right)}{4\ell_k^2\ell_{k'}^2}\left[x_S^0-(\underline{x}_S^0)_{kk'}-\frac{2i\ell_k^2\ell_{k'}^2(\underline{\lambda}_k-\underline{\lambda}_{k'})}{(\ell_k^2+\ell_{k'}^2)}\right]^2. \tag{36}$$

The second term yields an exponential fall off with $|\mathbf{y}_D-\mathbf{x}_S|^2$ in interference between neutrinos with different masses (and hence different velocities).[6] Setting Eqs. (33) and (36) into Eq. (28), integrating over x_S^0 and the unobserved external momenta, and dividing by dy_D^0 gives the expected event rate in the detector at time y_D^0.

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[6]Unlike the case of Eq. (18) in which there was a self-consistent way to assume a stationary phase, here it is necessary to

V. RELATIVISTIC LIMIT

Here the event rate in the limit of relativistic neutrinos will be exhibited. The zeroth order neutrino energy $\underline{\lambda}$ and coherence width ℓ are given by Eqs. (23) and (25) respectively, with $m_k = 0$. To first order,

$$\begin{aligned}
\underline{\lambda}_k &= \underline{\lambda} + \delta\underline{\lambda}_k, \\
s_k(\underline{\lambda}_k) &= \underline{\lambda} + \delta\underline{\lambda}_k - \frac{m_k^2}{2\underline{\lambda}}, \\
v_k &= 1 - \frac{m_k^2}{2\underline{\lambda}^2}, \\
\ell_k^2 &= \ell^2 + \delta\ell_k^2, \\
\underline{v}_{kk'} &= 1 - \frac{(m_k^2 + m_{k'}^2)}{4\underline{\lambda}^2},
\end{aligned} \tag{37}$$

where the explicit forms for $\delta\underline{\lambda}_k$ and $\delta\ell_k^2$ are determined by Eqs. (23) and (25) respectively. It will be assumed that m_k^2 can be neglected everywhere except when appearing with the macroscopic distance $|\mathbf{y}_D - \mathbf{x}_S|$ in the argument of the exponential, which magnifies its impact. This means that the third term in Eq. (36) can be neglected, and that

$$D_k(\underline{\lambda}_k) \approx D(\underline{\lambda}), \tag{38}$$

where $D(\underline{\lambda})$ is given by Eq. (19) with ξ_k replaced by $\xi \equiv \left(\underline{\lambda}, \underline{\lambda}\hat{\mathbf{L}}\right)$. It also means that

$$\begin{aligned}
\underline{\tilde{M}}_2 \left[\underline{\lambda}_k - s_k(\underline{\lambda}_k)\boldsymbol{\sigma}\cdot\hat{\mathbf{L}}\right] \underline{\tilde{M}}_1 &\approx \underline{\tilde{M}}_2 \left[\underline{\lambda} - \underline{\lambda}\boldsymbol{\sigma}\cdot\hat{\mathbf{L}}\right] \underline{\tilde{M}}_1 \\
&= \left[\mathcal{M}_S\left(\{\underline{\mathbf{k}}\}, \underline{\lambda}\right)\right] \left[\mathcal{M}_D\left(\{\underline{\mathbf{p}}\}, \underline{\lambda}\right)\right],
\end{aligned} \tag{39}$$

where $\mathcal{M}_S$ and $\mathcal{M}_D$ are the δ function-free matrix elements that would appear in the plane wave S-matrices [see Eq. (1)] describing the source and detector reactions with a massless neutrino of momentum $\underline{\lambda}\hat{\mathbf{L}}$, with the other particles having momenta $\{\underline{\mathbf{k}}\}$ (source) and $\{\underline{\mathbf{p}}\}$ (detector) [8]. The event rate at detector time y_D^0 is

$$\begin{aligned}
d\Gamma(y_D^0) = d\mathbf{P}' \int d\mathbf{x}_S \int d\mathbf{y}_D \int d\mathbf{K}(x_S)|_{x_S^0 = \underline{x}_S^0} \int d\mathbf{K}' \int d\mathbf{P}(y_D) \frac{8\mathcal{V}_S\mathcal{V}_D}{\pi^3 |\mathbf{y}_D - \mathbf{x}_S|^2} \\
\times \left[\sum_{\text{spins}} \left|\mathcal{M}_S\left(\{\underline{\mathbf{k}}\}, \underline{\lambda}\right)\right|^2\right] \left[\sum_{\text{spins}} \left|\mathcal{M}_D\left(\{\underline{\mathbf{p}}\}, \underline{\lambda}\right)\right|^2\right] \left(\frac{\pi}{2\ell^2}\right)^{1/2} e^{-2D(\underline{\lambda})} \\
\times \sum_{k,k'} U_{\alpha k} U_{\beta k}^* U_{\alpha k'}^* U_{\beta k'} \exp\left[-i\frac{(m_k^2 - m_{k'}^2)|\mathbf{y}_D - \mathbf{x}_S|}{2\underline{\lambda}} - \frac{(m_k^2 - m_{k'}^2)^2 |\mathbf{y}_D - \mathbf{x}_S|^2}{32\underline{\lambda}^4\ell^2}\right].
\end{aligned} \tag{40}$$

It can be shown that

$$\left(\frac{\pi}{2\ell^2}\right)^{1/2} e^{-2D(\underline{\lambda})} = \int dE_{\mathbf{q}}\, e^{-2D(E_{\mathbf{q}})}, \tag{41}$$

and consistency with the earlier approximate evaluation of the s^0 (or λ) integration means that $\left|\mathcal{M}_S\left(\{\underline{\mathbf{k}}\}, \underline{\lambda}\right)\right|^2$, $\left|\mathcal{M}_D\left(\{\underline{\mathbf{p}}\}, \underline{\lambda}\right)\right|^2$, and the factors summed over k, k' can be taken inside this integral as functions of $E_{\mathbf{q}}$ rather than $\underline{\lambda}$.

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include the phase in completing the square for x_S^0. It is this which gives rise to the second term in Eq. (36), which causes a loss of coherence with increasing $|\mathbf{y}_D - \mathbf{x}_S|^2$. It can be shown that the exponential fall off in $|\mathbf{y}_D - \mathbf{x}_S|^2$ also ensures that the phase $-\lambda(y_D^0 - x_S^0) + s_k(\lambda)|\mathbf{y}_D - \mathbf{x}_S|$ in Eq. (18) remains stationary over the range of λ for which $\exp[-D_k(\lambda)]$ is appreciably nonzero, for x_S^0 determined by the average velocity $\underline{v}_{kk'}$ [see Eq. (34)]. It will be recalled that the stationarity of this phase ensures that the minimum of $D_k(\lambda)$ dominates the integral (i.e., the "neutrino energy" becomes a meaningful concept).

In addition, if the phase space densities change little with energy variations and momentum variations of order ℓ^{-1}, then the leading contribution to Eq. (40) is the same as if the replacement

$$\begin{aligned} e^{-2D(E_{\mathbf{q}})} &\to \frac{\pi^4}{\sqrt{\mathrm{Det}\left[(W_S^{-1})_{\mu\nu}\right]}\sqrt{\mathrm{Det}\left[(W_D^{-1})_{\mu\nu}\right]}}\,\delta^4\left(-k_S+q\right)\delta^4\left(p_D-q\right) \\ &= \frac{\pi^8}{\mathcal{V}_S\mathcal{V}_D}\,\delta^4\left(-k_S+q\right)\delta^4\left(p_D-q\right) \end{aligned} \tag{42}$$

had been made, where $q=(E_{\mathbf{q}}, E_{\mathbf{q}}\,\hat{\mathbf{L}})$. Hence the leading contribution to the macroscopic event rate in the detector at time y_D^0 can be expressed

$$\begin{aligned} d\Gamma(y_D^0) = &\int d\mathbf{x}_S \int d\mathbf{y}_D \int \left[\prod_i^{I_S} \frac{d\underline{\mathbf{k}}_i}{(2\pi)^3}\right]\left[f\left(\underline{\mathbf{k}}_i, \mathbf{x}_S, x_S^0\right)\big|_{x_S^0=y_D^0-|\mathbf{y}_D-\mathbf{x}_S|}\right] \\ &\times \int \frac{d\underline{\mathbf{p}}}{(2\pi)^3}\, f\left(\underline{\mathbf{p}}, \mathbf{y}_D, y_D^0\right)\; d\Gamma\left(\{\underline{\mathbf{k}}\},\{\underline{\mathbf{p}}\}, \mathbf{x}_S, \mathbf{y}_D\right), \end{aligned} \tag{43}$$

where the single particle event rate is

$$\begin{aligned} d\Gamma\left(\{\underline{\mathbf{k}}\},\{\underline{\mathbf{p}}\}, \mathbf{x}_S, \mathbf{y}_D\right) \;=& \int dE_{\mathbf{q}}\left[\frac{d\Gamma\left(\{\underline{\mathbf{k}}\}, E_{\mathbf{q}}\right)}{|\mathbf{y}_D-\mathbf{x}_S|^2\, d\Omega_{\mathbf{q}}\, dE_{\mathbf{q}}}\right] \\ &\times \left[P_{\mathrm{mix}}\left(E_{\mathbf{q}}, \mathbf{x}_S, \mathbf{y}_D\right)\right]\left[d\sigma\left(\{\underline{\mathbf{p}}\}, E_{\mathbf{q}}\right)\right]. \end{aligned} \tag{44}$$

In Eq. (44),

$$\begin{aligned} dE_{\mathbf{q}}\left[\frac{d\Gamma\left(\{\underline{\mathbf{k}}\}, E_{\mathbf{q}}\right)}{|\mathbf{y}_D-\mathbf{x}_S|^2\, d\Omega_{\mathbf{q}}\, dE_{\mathbf{q}}}\right] \;=& \;\frac{1}{|\mathbf{y}_D-\mathbf{x}_S|^2}\frac{E_{\mathbf{q}}^2\, dE_{\mathbf{q}}}{(2\pi)^3(2E_{\mathbf{q}})}\left[\prod_i^{I_S}\frac{1}{\left(2E_{\underline{\mathbf{k}}_i}\right)}\right]\left[\prod_{i'}^{F_S}\int\frac{d\underline{\mathbf{k}}_{i'}}{(2\pi)^3\left(2E_{\underline{\mathbf{k}}_{i'}}\right)}\right] \\ &\times \sum_{\mathrm{spins}}\left|\mathcal{M}_S\left(\{\underline{\mathbf{k}}\}, E_{\mathbf{q}}\right)\right|^2\;(2\pi)^4\delta^4\left(-k_S+q\right) \end{aligned} \tag{45}$$

is the flux of neutrinos of energy $E_{\mathbf{q}}$ at position $\mathbf{y}_D$ due to an interaction at $\mathbf{x}_S$, as would be computed with standard plane wave methods;

$$\begin{aligned} d\sigma\left(\{\underline{\mathbf{p}}\}, E_{\mathbf{q}}\right) \;=& \;\frac{1}{(2E_{\mathbf{q}})\left(2E_{\underline{\mathbf{p}}}\right)}\left[\prod_{j'}^{F_D}\frac{d\underline{\mathbf{p}}_{j'}}{(2\pi)^3\left(2E_{\underline{\mathbf{p}}_{j'}}\right)}\right] \\ &\times\sum_{\mathrm{spins}}\left|\mathcal{M}_D\left(\{\underline{\mathbf{p}}\}, E_{\mathbf{q}}\right)\right|^2\;(2\pi)^4\delta^4\left(p_D-q\right) \end{aligned} \tag{46}$$

is the cross section for a massless neutrino interaction in the detector (assuming nonrelativistic initial state detector particle momentum $\underline{\mathbf{p}}$, so that the Møller velocity is equal to 1), and

$$\begin{aligned} P_{\mathrm{mix}}\left(E_{\mathbf{q}}, \mathbf{x}_S, \mathbf{y}_D\right) \;=& \sum_{k,k'} U_{\alpha k}U^*_{\beta k}U^*_{\alpha k'}U_{\beta k'} \\ &\times\exp\left[-i\frac{(m_k^2-m_{k'}^2)|\mathbf{y}_D-\mathbf{x}_S|}{2E_{\mathbf{q}}}-\frac{(m_k^2-m_{k'}^2)^2|\mathbf{y}_D-\mathbf{x}_S|^2}{32E_{\mathbf{q}}^4\ell^2}\right] \end{aligned} \tag{47}$$

is the flavor mixing (or "oscillation") probability.

Except for two differences, Equations (43)-(47) are just what one would write down for a macroscopic event rate using the naive QM model of the neutrino flavor mixing process, together with elementary considerations for the production flux and detection cross sections. The first difference is one that could also have been put in by hand,

namely, the causal time delay between emission and detection.[7] The second difference is the damping of coherence at very large distances, discussed earlier in this section.

VI. DISCUSSION

The calculations presented here concern the description of neutrino flavor mixing as a quantum field theoretic process, with the neutrinos as virtual particles connecting the "on shell" external particles involved in the neutrino production and detection reactions. This framework provides a more realistic description of and deeper physical insight into the flavor mixing process than the naive quantum mechanical model. Development beyond previous works has been sought in this study by considering general production and detection processes and wave packet functional forms, avoiding the unrealistic assumption of microscopic stationarity, leaving the time of the detection event (but not the emission event) an observable quantity, and making a complete connection to fully normalized event rates. The final results are given in Eqs. (43)-(47). Note that Eq. (47) has a damping factor for interference terms in addition to the usual oscillatory factor; this should formally be considered part of the "oscillation probability," which (without ambiguity, in the present fully normalized treatment) includes everything outside of the neutrino emission flux and detection cross section.

The free external particle wave packet picture is convenient for a number of reasons. Unlike descriptions involving bound states in the source, it is a suitable description for astrophysical neutrino sources such as the Sun (modulo the Coulomb repulsion of reacting nuclei, which can be suitably included) or supernovae. The lack of stationarity arises naturally due to the finite duration of the wave packets' overlap. The matter of normalization is simple since free particle states are employed. Furthermore, the dynamics are already built into the S-matrix, making the description of neutrino oscillations a matter of working out the kinematics.

This S-matrix framework could be generalized without much difficulty to include bound states for some of the source and detector particles. (The remaining particles would still be considered free particle wave packets. In this framework, the coherence times of the source and detector—which ultimately result from complicated microscopic many-body physics not considered here—can be considered as input parameters which ultimately manifest themselves in the finite free particle wave packet sizes.) The analogue of the S-matrix would be the amplitude for particular plane wave states to interact with particular bound states. A superposition of such amplitudes over several plane wave momenta—in order to create time-dependent wave packets for the external free particles—would constitute the amplitude for the overall neutrino production/propagation/detection process, with the neutrino production and detection localized in space *and* time. In going to the macroscopic rates, the square of bound state wave functions would be replaced by a sum over the phase space distribution of the relevant bound state quantum numbers.

While not new to this study, three basic insights into the neutrino flavor mixing process are listed here for completeness. First, *(1) an "oscillation probability" independent of the details of production and detection can only be defined in the relativstic limit.* This limit allows the neutrinos to become effectively on shell (i.e. massless) as far as production and detection are concerned. Assuming chiral interactions, the relativistic limit also causes only one neutrino spin to contribute, so that the overall squared amplitude $|\mathcal{M}|^2$ can factorize into separate production and detection squared amplitudes $|\mathcal{M}_S|^2$ and $|\mathcal{M}_D|^2$. This process has been shown in detail in this paper, culminating in Eqs. (43)-(47).

The second insight is a condition on *(2) the maximum size of the external particle coordinate space wave function overlap in the source (L_S) and detector (L_D) that allows neutrino states of different mass to interfere coherently:*

$$L_S,\ L_D \lesssim 0.2 \text{ m} \left(\frac{E_\nu}{\text{MeV}}\right)\left(\frac{\text{eV}^2}{|m_k^2 - m_{k'}^2|}\right) \equiv L_{\text{osc}}/(4\pi). \tag{48}$$

When expressed in terms of the "oscillation length" this condition is intuitively obvious. Its necessity can be seen mathematically in Eqs. (19) and (24). From these equations one can see that the "neutrino energy" is determined by a compromise between the degrees to which energy and momentum are conserved in the source and detector. However, the tendency towards energy conservation generally has a greater impact, i.e. the timelike components $(W_S^{-1})_{00} \sim (T_S)^2$ and $(W_D^{-1})_{00} \sim (T_D)^2$ (where $T_S,\ T_D$ are the time scales of the wave packet overlaps in the source

[7] In setting $x_S^0 = y_D^0 - |\mathbf{y}_D - \mathbf{x}_S|$ in Eq. (43), it has been assumed that the phase space densities f vary on time scales slower than 8×10^{-19} s $[(m_k^2\ +\ m_{k'}^2)\ /\ \text{eV}^2](\text{MeV}^2\ /\ E_{\mathbf{q}}^2)(|\mathbf{y}_D - \mathbf{x}_S|\ /\ \text{km})$, for all k and k', in order that the phase space densities could be taken out of the sum.

and detector) are larger than the spacelike $(W_S^{-1})_{ii} \sim (L_S)^2$ and $(W_D^{-1})_{ii} \sim (L_D)^2$ since the external particles travel slower than the speed of light. To the extent that the energy is more well determined, there must be a greater spread in momentum in order for interference to occur, which is why the condition expressed above is couched in terms of the spatial (as opposed to temporal) spread of the wave packet overlap.

The finite duration of the production and detection processes leads to the third insight. Contributions to the amplitude from neutrinos that deviate from a classical spacetime trajectory are exponentially suppressed [see Eqs. (24) and (27)]. This leads to *(3) an upper limit on the number of observable oscillations in space:*

$$N_{\rm osc} = \frac{|\mathbf{y}_D - \mathbf{x}_S|}{L_{\rm osc}} \lesssim \pi^{-1} E_\nu \ell \sim \frac{E_\nu}{\Delta E_\nu} \tag{49}$$

[see Eq. (47)]. Here the "coherence width" $\ell \sim \sqrt{(T_S)^2 + (T_D)^2 + (L_S)^2 + (L_D)^2}$ [see Eqs. (25) and (26)], and the detector resolution $\Delta E_\nu = \Delta p_D^0$ has been taken as a crude estimate of this quantity. Thus many oscillations in space should be visible before decoherence sets in as the spatial *and* temporal[8] resolution of the detector begins to distinguish the separating neutrino mass eigenstates.

It is true that the three insights above can, to some extent, be achieved without the elaborate machinery presented here. As far as insight (1) goes, common knowledge that weak interactions are $V - A$ makes the irrelevance of the neutrino spin degree of freedom in the relativistic limit somewhat obvious. The relativistic limit also makes the notion of "real" flavor eigenstates reasonable (zero mass is on-shell). One can then adopt the simplified quantum mechanical picture. In connection with this simplified quantum mechanical picture, condition (2) can be argued from the uncertainty principle, $\Delta x \Delta p \gtrsim 1$, where Δx corresponds to the oscillation length $L_{\rm osc}$ and Δp corresponds to the inverse source/detector sizes $L_{S,D}^{-1}$ [14]. Condition (3) follows from noting that a real source (detector) will have some finite linewidth (resolution). In order that interference terms not wash out when binned over this energy range ΔE, it is necessary (e.g., [10]) that the variation in the oscillation phase, $\Delta[(m_k^2 - m_{k'}^2)|\mathbf{y}_D - \mathbf{x}_S|/E] \sim (|\mathbf{y}_D - \mathbf{y}_S|/L_{\rm osc})(\Delta E/E)$, be smaller than 2π, which is essentially condition (3).

Even if these arguments can be made in some fashion in connection with the simplistic formalism, the whole picture lends itself to conceptual difficulties at some level [9]. Furthermore, the conditions (1)-(3) must be invoked from principles outside the formalism itself. In contrast, the QFT description of the neutrino production/propagation/detection presented here exhibits all of these conditions in a natural, self-contained, and physically unambiguous manner, and describes the transition to the failure of these conditions in a quantitative way.

ACKNOWLEDGMENTS

It is a pleasure to thank Daniel J. H. Chung for instructive discussions and patient collaboration. Discussions with John Beacom and Georg Raffelt are also gratefully acknowledged. This work was completed at the GAAC sponsored meeting held at the Max Plank Institute for Physics and Max Plank Institute for Astrophysics. CYC is supported by DOE grant FG02-87ER40317.

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[1] C. Giunti, C. W. Kim, J. A. Lee, and U. W. Lee, Phys. Rev. D **48**, 4310 (1993).
[2] W. Grimus and P. Stockinger, Phys. Rev. D **54**, 3414 (1996).
[3] J. E. Campagne, Phys. Lett. **400B**, 135 (1997).
[4] K. Kiers and N. Weiss, Phys. Rev. D **57**, 4418 (1998).
[5] C. Giunti, C. W. Kim, and U. W. Lee, Phys. Lett. **421B**, 237 (1998).
[6] W. Grimus and P. Stockinger, Phys. Rev. D **59**, 013011 (1998).
[7] A. Ioannisian and A. Pilaftsis, Phys. Rev. D **59**, 053003 (1999).
[8] C. Y. Cardall and D. J. H. Chung, Phys. Rev. D **60**, 073012 (1999).

[8]Note that if T_S or $T_D \to \infty$, coherence is restored for infinite propagation distances. This dependence on the coherence time in addition to the spatial extent of the detector was noted explicitly in Refs. [4] and [5], though it was implicitly present in Ref. [1] as well.

[9] J. Rich, Phys. Rev. D **48**, 4318 (1993).
[10] L. Stodolsky, Phys. Rev. D **58**, 036006 (1999).
[11] M. E. Peskin and D. V. Schroeder, *Introduction to Quantum Field Theory* (Reading, Addison-Wesley, 1995).
[12] M. L. Goldberger and K. M. Watson, *Collision Theory* (New York, John Wiley & Sons, 1964).
[13] A. B. Balantekin and J. F. Beacom, Phys. Rev. D **54**, 6323 (1996).
[14] B. Kayser, Phys. Rev. D **24**, 110 (1981).